

RATIONALITY OF SPECTRAL ACTION FOR ROBERTSON-WALKER METRICS

FARZAD FATHIZADEH, ASGHAR GHORBANPOUR, MASOUD KHALKHALI

Department of Mathematics, Western University
London, Ontario, Canada, N6A 5B7 ¹

ABSTRACT. We use pseudodifferential calculus and heat kernel techniques to prove a conjecture by Chamseddine and Connes on rationality of the coefficients of the polynomials in the cosmic scale factor $a(t)$ and its higher derivatives, which describe the general terms a_{2n} in the expansion of the spectral action for general Robertson-Walker metrics. We also compute the terms up to a_{12} in the expansion of the spectral action by our method. As a byproduct, we verify that our computations agree with the terms up to a_{10} that were previously computed by Chamseddine and Connes by a different method.

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¹*E-mail addresses:* ffathiz@uwo.ca, aghorba@uwo.ca, masoud@uwo.ca

1. INTRODUCTION

Noncommutative geometry in the sense of Alain Connes [11] has provided a paradigm for geometry in the noncommutative setting based on spectral data. This generalizes Riemannian geometry [14] and incorporates physical models of elementary particle physics [12, 13, 10, 15, 5, 7, 19, 32, 33, 34]. An outstanding feature of the spectral action defined for noncommutative geometries is that it derives the Lagrangian of the physical models from simple noncommutative geometric data [13, 4, 10]. Thus various methods have been developed for computing the terms in the expansion in the energy scale Λ of the spectral action [3, 6, 8, 9, 20, 21]. Potential applications of noncommutative geometry in cosmology have recently been carried out in [22, 25, 26, 27, 28, 29, 30, 31, 16].

Noncommutative geometric spaces are described by spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where \mathcal{A} is an involutive algebra represented by bounded operators on a Hilbert space \mathcal{H} , and D is an unbounded self-adjoint operator acting in \mathcal{H} [11]. The operator D , which plays the role of the Dirac operator, encodes the metric information and it is further assumed that it has bounded commutators with elements of \mathcal{A} . It has been shown that if \mathcal{A} is commutative and the triple satisfies suitable regularity conditions then \mathcal{A} is the algebra of smooth functions on a spin^c manifold M and D is the Dirac operator acting in the Hilbert space of L^2 -spinors [14]. In this case, the Seeley-de Witt coefficients $a_n(D^2) = \int_M a_n(x, D^2) dv(x)$, which vanish for odd n , appear in a small time asymptotic expansion of the form

$$\text{Tr}(e^{-tD^2}) \sim t^{-\dim(M)/2} \sum_{n \geq 0} a_{2n}(D^2) t^n \quad (t \rightarrow 0).$$

These coefficients determine the terms in the expansion of the spectral action. That is, there is an expansion of the form

$$\text{Tr} f(D^2/\Lambda^2) \sim \sum_{n \geq 0} f_{2n} a_{2n}(D^2/\Lambda^2),$$

where f is a positive even function defined on the real line, and f_{2n} are the moments of the function f [4, 3]. See Theorem 1.145 in [15] for details in a more general setup, namely for spectral triples with simple dimension spectrum.

By devising a direct method based on the Euler-Maclaurin formula and the Feynman-Kac formula, Chamseddine and Connes have initiated in [9] a detailed study of the spectral action for the Robertson-Walker metric with a general cosmic scale factor $a(t)$. They calculated the terms up to a_{10} in the expansion and checked the agreement of the terms up to a_6 against Gilkey's universal formulas [17, 18].

The present paper is intended to compute the term a_{12} in the spectral action for general Robertson-Walker metrics, and to prove the conjecture of Chamseddine and Connes [9] on rationality of the coefficients of the polynomials in $a(t)$ and its derivatives that describe the general terms a_{2n} in the expansion. In passing, we compare the outcome of our computations up to the term a_{10} with the expressions obtained in [9], and confirm their agreement.

In terms of the above aims, explicit formulas for the Dirac operator of the Robertson-Walker metric and its pseudodifferential symbol in Hopf coordinates are derived in §2. Following a brief review of the heat kernel method for computing local invariants of elliptic differential operators using pseudodifferential calculus [17], we compute in §3 the terms up to a_{10} in the expansion of the spectral action

for Robertson-Walker metrics. The outcome of our calculations confirms the expressions obtained in [9]. This forms a check in particular on the validity of a_8 and a_{10} , which as suggested in [9] also, seems necessary due to the high complexity of the formulas. In §4, we record the expression for the term a_{12} achieved by a significantly heavier computation, compared to the previous terms. It is checked that the reduction of a_{12} to the round case $a(t) = \sin t$ conforms to the full expansion obtained in [9] for the round metric by remarkable calculations that are based on the Euler-Maclaurin formula. In order to validate our expression for a_{12} , parallel but completely different computations are performed in spherical coordinates and the final results are confirmed to match precisely with our calculations in Hopf coordinates.

In §5, we prove the conjecture made in [9] on rationality of the coefficients appearing in the expressions for the terms of the spectral action for Robertson-Walker metrics. That is, we show that the term a_{2n} in the expansion is of the form $Q_{2n}(a(t), a'(t), \dots, a^{(2n)}(t))/a(t)^{2n-3}$, where Q_{2n} is a polynomial with rational coefficients. We also find a formula for the coefficient of the term with the highest derivate of $a(t)$ in a_{2n} . It is known that values of Feynman integrals for quantum gauge theories are closely related to multiple zeta values and periods in general and hence tend to be transcendental numbers [24]. In sharp distinction, the rationality result proved in this paper is valid for all scale factors $a(t)$ in Robertson-Walker metrics. Although it might be exceedingly difficult, it is certainly desirable to find all the terms a_{2n} in the spectral action. The rationality result is a consequence of a certain symmetry in the heat kernel and it is plausible that this symmetry would eventually reveal the full structure of the coefficients a_{2n} . This is a task for a future work. Our main conclusions are summarized in §6.

2. THE DIRAC OPERATOR FOR ROBERTSON-WALKER METRICS

According to the spectral action principle [12, 4], the spectral action of any geometry depends on its Dirac operator since the terms in the expansion are determined by the high frequency behavior of the eigenvalues of this operator. For spin manifolds, the explicit computation of the Dirac operator in a coordinate system is most efficiently achieved by writing its formula after lifting the Levi-Civita connection on the cotangent bundle to the spin connection on the spin bundle. In this section, we summarize this formalism and compute the Dirac operator of the Robertson-Walker metric in Hopf coordinates. Throughout this paper we use Einstein's summation convention without any further notice.

2.1. Levi-Civita connection. The spin connection of any spin manifold M is the lift of the Levi-Civita connection for the cotangent bundle T^*M to the spin bundle. Let us, therefore, recall the following recipe for computing the Levi-Civita connection and thereby the spin connection of M . Given an orthonormal frame $\{\theta_\alpha\}$ for the tangent bundle TM and its dual coframe $\{\theta^\alpha\}$, the connection 1-forms ω_β^α of any connection ∇ on T^*M are defined by

$$\nabla \theta^\alpha = \omega_\beta^\alpha \theta^\beta.$$

Since the Levi-Civita connection is the unique torsion free connection which is compatible with the metric, its 1-forms are uniquely determined by

$$d\theta^\beta = \omega_\alpha^\beta \wedge \theta^\alpha.$$

This is justified by the fact that the compatibility with metric enforces the relations

$$\omega_\beta^\alpha = -\omega_\alpha^\beta,$$

while, taking advantage of the first Cartan structure equation, the torsion-freeness amounts to the vanishing of

$$T^\alpha = d\theta^\alpha - \omega_\beta^\alpha \wedge \theta^\beta.$$

2.2. The spin connection of Robertson-Walker metrics in Hopf coordinates. The (Euclidean) Robertson-Walker metric with the cosmic scale factor $a(t)$ is given by

$$ds^2 = dt^2 + a^2(t) d\sigma^2,$$

where $d\sigma^2$ is the round metric on the 3-sphere \mathbb{S}^3 . It is customary to write this metric in spherical coordinates, however, for our purposes which will be explained below, it is more convenient to use the Hopf coordinates, which parametrize the 3-sphere $S^3 \subset \mathbb{C}^2$ by

$$z_1 = e^{i\phi_1} \sin(\eta), \quad z_2 = e^{i\phi_2} \cos(\eta),$$

with η ranging in $[0, \pi/2)$ and ϕ_1, ϕ_2 ranging in $[0, 2\pi)$. The Robertson-Walker metric in the coordinate system $x = (t, \eta, \phi_1, \phi_2)$ is thus given by

$$ds^2 = dt^2 + a^2(t) (d\eta^2 + \sin^2(\eta) d\phi_1^2 + \cos^2(\eta) d\phi_2^2).$$

An orthonormal coframe for ds^2 is then provided by

$$\theta^1 = dt, \quad \theta^2 = a(t) d\eta, \quad \theta^3 = a(t) \sin \eta d\phi_1, \quad \theta^4 = a(t) \cos \eta d\phi_2.$$

Applying the exterior derivative to these forms, one can easily show that they satisfy the following equations, which determine the connection 1-forms of the Levi-Civita connection:

$$\begin{aligned} d\theta^1 &= 0, \\ d\theta^2 &= \frac{a'(t)}{a(t)} \theta^1 \wedge \theta^2, \\ d\theta^3 &= \frac{a'(t)}{a(t)} \theta^1 \wedge \theta^3 + \frac{\cot \eta}{a(t)} \theta^2 \wedge \theta^3, \\ d\theta^4 &= \frac{a'(t)}{a(t)} \theta^1 \wedge \theta^4 - \frac{\tan \eta}{a(t)} \theta^2 \wedge \theta^4. \end{aligned}$$

We recast the above equations into the matrix of connection 1-forms

$$\omega = \frac{1}{a(t)} \begin{pmatrix} 0 & -a'(t) \theta^2 & -a'(t) \theta^3 & -a'(t) \theta^4 \\ a'(t) \theta^2 & 0 & -\cot \eta \theta^3 & \tan \eta \theta^4 \\ a'(t) \theta^3 & \cot \eta \theta^3 & 0 & 0 \\ a'(t) \theta^4 & -\tan \eta \theta^4 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(4),$$

which lifts to the spin bundle using the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \rightarrow \mathfrak{spin}(4)$ given by (see [23])

$$\mu(A) = \frac{1}{4} \sum_{\alpha, \beta} \langle A\theta^\alpha, \theta^\beta \rangle c(\theta^\alpha) c(\theta^\beta), \quad A \in \mathfrak{so}(4).$$

Since $\langle \omega\theta^\alpha, \theta^\beta \rangle = \omega_\beta^\alpha$, the lifted connection $\tilde{\omega}$ is written as

$$\tilde{\omega} = \frac{1}{4} \sum_{\alpha, \beta} \omega_\beta^\alpha c(\theta^\alpha) c(\theta^\beta).$$

In the case of the Robertson-Walker metric we find that

$$(1) \quad \tilde{\omega} = \frac{1}{2a(t)} (a'(t)\theta^2\gamma^{12} + a'(t)\theta^3\gamma^{13} + a'(t)\theta^4\gamma^{14} + \cot(\eta)\theta^3\gamma^{23} - \tan(\eta)\theta^4\gamma^{24}),$$

where we use the notation $\gamma^{ij} = \gamma^i\gamma^j$ for products of pairs of the gamma matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$, which are respectively written as

$$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

2.3. The Dirac Operator of Robertson-Walker metrics in Hopf coordinates. Using the expression (1) obtained for the spin connection and considering the predual of the orthonormal coframe $\{\theta^\alpha\}$,

$$\theta_1 = \frac{\partial}{\partial t}, \quad \theta_2 = \frac{1}{a(t)} \frac{\partial}{\partial \eta}, \quad \theta_3 = \frac{1}{a(t) \sin \eta} \frac{\partial}{\partial \phi_1}, \quad \theta_4 = \frac{1}{a(t) \cos \eta} \frac{\partial}{\partial \phi_2},$$

we compute the Dirac operator for the Robertson-Walker metric explicitly:

$$\begin{aligned} D &= c(\theta^\alpha) \nabla_{\theta_\alpha} \\ &= \gamma^\alpha (\theta_\alpha + \tilde{\omega}(\theta_\alpha)) \\ &= \gamma^1 \left(\frac{\partial}{\partial t} \right) + \gamma^2 \left(\frac{1}{a} \frac{\partial}{\partial \eta} + \frac{a'}{2a} \gamma^{12} \right) + \gamma^3 \left(\frac{1}{a \sin(\eta)} \frac{\partial}{\partial \phi_1} + \frac{a'}{2a} \gamma^{13} + \frac{\cot(\eta)}{2a} \gamma^{23} \right) \\ &\quad + \gamma^4 \left(\frac{1}{a \cos(\eta)} \frac{\partial}{\partial \phi_2} + \frac{a'}{2a} \gamma^{14} - \frac{\tan(\eta)}{2a} \gamma^{24} \right) \\ &= \gamma^1 \frac{\partial}{\partial t} + \gamma^2 \frac{1}{a} \frac{\partial}{\partial \eta} + \gamma^3 \frac{1}{a \sin \eta} \frac{\partial}{\partial \phi_1} + \gamma^4 \frac{1}{a \cos \eta} \frac{\partial}{\partial \phi_2} + \frac{3a'}{2a} \gamma^1 + \frac{\cot(2\eta)}{a} \gamma^2. \end{aligned}$$

Thus the pseudodifferential symbol of D is given by

$$\sigma_D(x, \xi) = i\xi_1\gamma^1 + \frac{i\xi_2}{a}\gamma^2 + \frac{i\xi_3}{a \sin \eta}\gamma^3 + \frac{i\xi_4}{a \cos \eta}\gamma^4 + \frac{3a'}{2a}\gamma^1 + \frac{\cot(2\eta)}{a}\gamma^2.$$

For the purpose of employing pseudodifferential calculus in the sequel to compute the heat coefficients, we record in the following proposition the pseudodifferential symbol of D^2 . This can be achieved by a straightforward computation to find an explicit expression for D^2 , or alternatively, one can apply the composition rule for symbols, $\sigma_{P_1 P_2}(x, \xi) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{P_1} \partial_x^\alpha \sigma_{P_2}$, to the symbol of D .

Proposition 2.1. *The pseudodifferential symbol of D^2 , where D is the Dirac operator for the Robertson-Walker metric, is given by*

$$\sigma(D^2) = p_2 + p_1 + p_0,$$

where the homogeneous components p_i of order i are written as

$$\begin{aligned}
p_2 &= \xi_1^2 + \frac{1}{a^2} \xi_2^2 + \frac{1}{a^2 \sin^2(\eta)} \xi_3^2 + \frac{1}{a^2 \cos^2(\eta)} \xi_4^2, \\
p_1 &= \frac{-3iaa'}{a^2} \xi_1 + \frac{-ia'\gamma^{12} - 2i \cot(2\eta)}{a^2} \xi_2 - \frac{ia' \csc(\eta) \gamma^{13} + i \cot(\eta) \csc(\eta) \gamma^{23}}{a^2} \xi_3 \\
&\quad + \frac{i \tan(\eta) \sec(\eta) \gamma^{24} - ia' \sec(\eta) \gamma^{14}}{a^2} \xi_4, \\
p_0 &= \frac{1}{4a(t)^2} \left(-6a(t)a''(t) - 3a'(t)^2 + \csc^2(\eta) + \sec^2(\eta) \right. \\
(2) \quad &\quad \left. + 4 + 2a'(t)(\cot(\eta) - \tan(\eta))\gamma^{12} \right).
\end{aligned}$$

3. TERMS UP TO a_{10} AND THEIR AGREEMENT WITH CHAMSEDDINE-CONNES' RESULT

The computation of the terms in the expansion of the spectral action for a spin manifold, or equivalently the calculation of the heat coefficients, can be achieved by recursive formulas while working in the heat kernel scheme of local invariants of elliptic differential operators and index theory [17]. Pseudodifferential calculus is an effective tool for dealing with the necessary approximations for deriving the small time asymptotic expansions in which the heat coefficients appear. Universal formulas in terms of the Riemann curvature operator and its contractions and covariant derivatives are written in the literature only for the terms up to a_{10} , namely Gilkey's formulas up to a_6 [17, 18] and the formulas in [1, 2, 35] for a_8 and a_{10} .

3.1. Small time heat kernel expansions using pseudodifferential calculus.

In [17], by appealing to the Cauchy integral formula and using pseudodifferential calculus, recursive formulas for the heat coefficients of elliptic differential operators are derived. That is, one writes ²

$$e^{-tD^2} = -\frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} (D^2 - \lambda)^{-1} d\lambda,$$

where the contour γ goes around the non-negative real axis in the counterclockwise direction, and one uses pseudodifferential calculus to approximate $(D^2 - \lambda)^{-1}$ via the homogeneous terms appearing in the expansion of the symbol of the parametrix of $D^2 - \lambda$. Although left and right parametrices have the same homogeneous components, for the purpose of finding recursive formulas for the coefficients appearing in each component, which will be explained shortly, it is more convenient for us to consider the right parametrix $\tilde{R}(\lambda)$. Therefore, the next task is to compute recursively the homogeneous pseudodifferential symbols r_j of order $-2 - j$ in the expansion of $\sigma(\tilde{R}(\lambda))$. Using the calculus of symbols, with the crucial nuance that λ is considered to be of order 2, one finds that

$$r_0 = (p_2 - \lambda)^{-1},$$

²Hereafter in this paper t denotes the first variable of the space when it appears in $a(t)$ and its derivatives and it denotes the time when it appears in the heat operator and the associated small time asymptotic expansions.

and for any $n > 1$

$$(3) \quad r_n = -r_0 \sum_{\substack{|\alpha| + j + 2 - k = n \\ j < n}} \frac{(-i)^{|\alpha|}}{\alpha!} d_\xi^\alpha p_k d_x^\alpha r_j.$$

We summarize the process of obtaining the heat coefficients by explaining that one then uses these homogeneous terms in the Cauchy integral formula to approximate the integral kernel of e^{-tD^2} . Integration of the kernel of this operator on the diagonal yields a small time asymptotic expansion of the form

$$\text{Tr}(e^{-tD^2}) \sim \sum_{n=0}^{\infty} \frac{t^{(n-4)/2}}{16\pi^4} \int \text{tr}(e_n(x)) d\text{vol}_g \quad (t \rightarrow 0),$$

where

$$(4) \quad e_n(x) \sqrt{\det g} = \frac{-1}{2\pi i} \int \int_\gamma e^{-\lambda} r_n(x, \xi, \lambda) d\lambda d\xi.$$

For detailed discussions, we refer the reader to [17].

It is clear from (2) that cross derivatives of p_2 vanish and $d_\xi^\alpha p_k = 0$ if $|\alpha| > k$. Furthermore, $\frac{\partial}{\partial \phi_k} r_n = 0$ for $n \geq 0$, and the summation (3) is written as

$$(5) \quad \begin{aligned} r_n = & -r_0 p_0 r_{n-2} - r_0 p_1 r_{n-1} + i r_0 \frac{\partial}{\partial \xi_1} p_1 \frac{\partial}{\partial t} r_{n-2} + i r_0 \frac{\partial}{\partial \xi_2} p_1 \frac{\partial}{\partial \eta} r_{n-2} \\ & + i r_0 \frac{\partial}{\partial \xi_1} p_2 \frac{\partial}{\partial t} r_{n-1} + i r_0 \frac{\partial}{\partial \xi_2} p_2 \frac{\partial}{\partial \eta} r_{n-1} + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_1^2} p_2 \frac{\partial^2}{\partial t^2} r_{n-2} \\ & + \frac{1}{2} r_0 \frac{\partial^2}{\partial \xi_2^2} p_2 \frac{\partial^2}{\partial \eta^2} r_{n-2}. \end{aligned}$$

Using induction, we find that

$$(6) \quad r_n = \sum_{\substack{2j-2-|\alpha|=n \\ n/2+1 \leq j \leq 2n+1}} r_{n,j,\alpha}(x) r_0^j \xi^\alpha.$$

For example, one can see that for $n = 0$ the only non-zero $r_{0,j,\alpha}$ is $r_{0,1,0} = 1$, and for $n = 1$ the non-vanishing terms are

$$r_{1,2,\mathbf{e}_k} = \frac{\partial p_1}{\partial \xi_k}, \quad r_{1,3,2\mathbf{e}_l+\mathbf{e}_k} = -2ig^{kk} \frac{\partial g^{ll}}{\partial x_k},$$

where \mathbf{e}_j denotes the j -th standard unit vector in \mathbb{R}^4 .

It then follows from the equations (4), (5) and (6) that

$$(7) \quad \begin{aligned} e_n(x) a(t)^3 \sin(\eta) \cos(\eta) &= \frac{-1}{2\pi i} \int_{\mathbb{R}^4} \int_\gamma e^{-t\lambda} r_n(x, \xi, \lambda) d\lambda d\xi \\ &= \sum r_{n,j,\alpha}(x) \int_{\mathbb{R}^4} \xi^\alpha \frac{-1}{2\pi i} \int_\gamma e^{-t\lambda} r_0^j d\lambda d\xi \\ &= \sum \frac{c_\alpha}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4+3} \sin(\eta)^{\alpha_3+1} \cos(\eta)^{\alpha_4+1}, \end{aligned}$$

where

$$c_\alpha = \prod_k \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k}+1}{2}.$$

It is straightforward to justify the latter using these identities:

$$\begin{aligned}\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} r_0^j d\lambda &= (-1)^j \frac{(-1)^{j-1}}{(j-1)!} e^{-\|\xi\|^2} = \frac{-1}{(j-1)!} \prod_{k=1}^4 e^{-g^{kk} \xi_k^2}, \\ \int_{\mathbb{R}} x^n e^{-bx^2} dx &= \frac{1}{2} ((-1)^n + 1) b^{-\frac{n}{2} - \frac{1}{2}} \Gamma\left(\frac{n+1}{2}\right).\end{aligned}$$

A key point that facilitates our calculations and the proof of our main theorem presented in §5.1 is the derivation of recursive formulas for the coefficients $r_{n,j,\alpha}$ as follows. By substitution of (6) into (5) we find a recursive formula of the form

$$\begin{aligned}(8) \quad r_{n,j,\alpha} &= -p_0 r_{n-2,j-1,\alpha} - \sum_k \frac{\partial p_1}{\partial \xi_k} r_{n-1,j-1,\alpha - \mathbf{e}_k} \\ &+ i \sum_k \frac{\partial p_1}{\partial \xi_k} \frac{\partial}{\partial x_k} r_{n-2,j-1,\alpha} + i(2-j) \sum_{k,l} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial p_1}{\partial \xi_k} r_{n-2,j-2,\alpha - 2\mathbf{e}_l} \\ &+ 2i \sum_k g^{kk} \frac{\partial}{\partial x_k} r_{n-1,j-1,\alpha - \mathbf{e}_k} + i(4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} r_{n-1,j-2,\alpha - 2\mathbf{e}_l - \mathbf{e}_k} \\ &+ \sum_k g^{kk} \frac{\partial^2}{\partial x_k^2} r_{n-2,j-1,\alpha} + (4-2j) \sum_{k,l} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial}{\partial x_k} r_{n-2,j-2,\alpha - 2\mathbf{e}_l} \\ &+ (2-j) \sum_{k,l} g^{kk} \frac{\partial^2 g^{ll}}{\partial x_k^2} r_{n-2,j-2,\alpha - 2\mathbf{e}_l} \\ &+ (3-j)(2-j) \sum_{k,l,l'} g^{kk} \frac{\partial g^{ll}}{\partial x_k} \frac{\partial g^{l'l'}}{\partial x_k} r_{n-2,j-3,\alpha - 2\mathbf{e}_l - 2\mathbf{e}_{l'}}.\end{aligned}$$

It is undeniable that the mechanism described above for computing the heat coefficients involves heavy computations which need to be overcome by computer programming. Calculating explicitly the functions $e_n(x)$, $n = 0, 2, \dots, 12$, and computing their integrals over \mathbb{S}_a^3 with computer assistance, we find the explicit polynomials in $a(t)$ and its derivatives recorded in the sequel, which describe the corresponding terms in the expansion of the spectral action for the Robertson-Walker metric. That is, each function a_n recorded below is the outcome of

$$\begin{aligned}a_n &= \frac{1}{16\pi^4} \int_{\mathbb{S}_a^3} \text{tr}(e_n) d\text{vol}_g \\ &= \frac{1}{16\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \text{tr}(e_n) a(t)^3 \sin(\eta) \cos(\eta) d\eta d\phi_1 d\phi_2.\end{aligned}$$

3.2. The terms up to a_6 . These terms were computed in [9] by their direct method, which is based on the Euler-Maclaurin summation formula and the Feynman-Kac formula, and they were checked by Gilkey's universal formulas. Our computations based on the method explained in the previous subsection also gives the same result.

The first term, whose integral up to a universal factor gives the volume, is given by

$$a_0 = \frac{a(t)^3}{2}.$$

Since the latter appears as the leading term in the small time asymptotic expansion of the heat kernel it is related to Weyl's law, which reads the volume from the asymptotic distribution of the eigenvalues of D^2 . The next term, which is related to the scalar curvature, has the expression

$$a_2 = \frac{1}{4}a(t) (a(t)a''(t) + a'(t)^2 - 1).$$

The term after, whose integral is topological, is related to the Gauss-Bonnet term (cf. [9]) and is written as

$$a_4 = \frac{1}{120} \left(3a^{(4)}(t)a(t)^2 + 3a(t)a''(t)^2 - 5a''(t) + 9a^{(3)}(t)a(t)a'(t) - 4a'(t)^2a''(t) \right).$$

The term a_6 , which is the last term for which Gilkey's universal formulas are written, is given by

$$a_6 = \frac{1}{5040a(t)^2} \left(9a^{(6)}(t)a(t)^4 - 21a^{(4)}(t)a(t)^2 - 3a^{(3)}(t)^2a(t)^3 - 56a(t)^2a''(t)^3 + 42a(t)a''(t)^2 + 36a^{(5)}(t)a(t)^3a'(t) + 6a^{(4)}(t)a(t)^3a''(t) - 42a^{(4)}(t)a(t)^2a'(t)^2 + 60a^{(3)}(t)a(t)a'(t)^3 + 21a^{(3)}(t)a(t)a'(t) + 240a(t)a'(t)^2a''(t)^2 - 60a'(t)^4a''(t) - 21a'(t)^2a''(t) - 252a^{(3)}(t)a(t)^2a'(t)a''(t) \right).$$

3.3. The terms a_8 and a_{10} . These terms were computed by Chamseddine and Connes in [9] using their direct method. In order to form a check on the final formulas, they have suggested to use the universal formulas of [1, 2, 35] to calculate these terms and compare the results. As mentioned earlier, Gilkey's universal formulas were used in [9] to check the terms up to a_6 , however, they are written in the literature only up to a_6 and become rather complicated even for this term.

In this subsection, we pursue the computation of the terms a_8 and a_{10} in the expansion of the spectral action for Robertson-Walker metrics by continuing to employ pseudodifferential calculus, as presented in §3.1, and check that the final formulas agree with the result in [9]. The final formulas for a_8 and a_{10} are the following expressions:

$$a_8 = -\frac{1}{10080a(t)^4} \left(-a^{(8)}(t)a(t)^6 + 3a^{(6)}(t)a(t)^4 + 13a^{(4)}(t)^2a(t)^5 - 24a^{(3)}(t)^2a(t)^3 - 114a(t)^3a''(t)^4 + 43a(t)^2a''(t)^3 - 5a^{(7)}(t)a(t)^5a'(t) + 2a^{(6)}(t)a(t)^5a''(t) + 9a^{(6)}(t)a(t)^4a'(t)^2 + 16a^{(3)}(t)a^{(5)}(t)a(t)^5 - 24a^{(5)}(t)a(t)^3a'(t)^3 - 6a^{(5)}(t)a(t)^3a'(t) + 69a^{(4)}(t)a(t)^4a''(t)^2 - 36a^{(4)}(t)a(t)^3a''(t) + 60a^{(4)}(t)a(t)^2a'(t)^4 + 15a^{(4)}(t)a(t)^2a'(t)^2 + 90a^{(3)}(t)^2a(t)^4a''(t) - 216a^{(3)}(t)^2a(t)^3a'(t)^2 - 108a^{(3)}(t)a(t)a'(t)^5 - 27a^{(3)}(t)a(t)a'(t)^3 + 801a(t)^2a'(t)^2a''(t)^3 - 588a(t)a'(t)^4a''(t)^2 - 87a(t)a'(t)^2a''(t)^2 + 108a'(t)^6a''(t) + 27a'(t)^4a''(t) + 78a^{(5)}(t)a(t)^4a'(t)a''(t) + 132a^{(3)}(t)a^{(4)}(t)a(t)^4a'(t) - 312a^{(4)}(t)a(t)^3a'(t)^2a''(t) - 819a^{(3)}(t)a(t)^3a'(t)a''(t)^2 + 768a^{(3)}(t)a(t)^2a'(t)^3a''(t) + 102a^{(3)}(t)a(t)^2a'(t)a''(t) \right),$$

and

$$a_{10} = \frac{1}{665280a(t)^6} \left(3a^{(10)}(t)a(t)^8 - 222a^{(5)}(t)^2a(t)^7 - 348a^{(4)}(t)a^{(6)}(t)a(t)^7 - 147a^{(3)}(t)a^{(7)}(t)a(t)^7 - 18a''(t)a^{(8)}(t)a(t)^7 + 18a'(t)a^{(9)}(t)a(t)^7 - 482a''(t)a^{(4)}(t)^2a(t)^6 - 331a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 1110a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 1556a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - 448a''(t)^2a^{(6)}(t)a(t)^6 - 1074a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - 476a'(t)a''(t)a^{(7)}(t)a(t)^6 - 43a'(t)^2a^{(8)}(t)a(t)^6 - 11a^{(8)}(t)a(t)^6 + 8943a'(t)a^{(3)}(t)^3a(t)^5 + 21846a''(t)^2a^{(3)}(t)^2a(t)^5 + 4092a'(t)^2a^{(4)}(t)^2a(t)^5 + 396a^{(4)}(t)^2a(t)^5 + 10560a''(t)^3a^{(4)}(t)a(t)^5 +$$

$$\begin{aligned}
& 39402a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 11352a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 6336a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + 594a^{(3)}(t)a^{(5)}(t)a(t)^5 + 2904a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + \\
& 264a''(t)a^{(6)}(t)a(t)^5 + 165a'(t)^3a^{(7)}(t)a(t)^5 + 33a'(t)a^{(7)}(t)a(t)^5 - 10338a''(t)^5a(t)^4 - \\
& 95919a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - 3729a''(t)a^{(3)}(t)^2a(t)^4 - 117600a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - \\
& 68664a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - 2772a''(t)^2a^{(4)}(t)a(t)^4 - 23976a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 2640a'(t)a^{(3)}(t)a^{(4)}(t)a(t)^4 - 12762a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 1386a'(t)a''(t)a^{(5)}(t)a(t)^4 - \\
& 651a'(t)^4a^{(6)}(t)a(t)^4 - 132a'(t)^2a^{(6)}(t)a(t)^4 + 111378a'(t)^2a''(t)^4a(t)^3 + 2354a''(t)^4a(t)^3 + \\
& 31344a'(t)^4a^{(3)}(t)^2a(t)^3 + 3729a'(t)^2a^{(3)}(t)^2a(t)^3 + 236706a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + \\
& 13926a'(t)a''(t)^2a^{(3)}(t)a(t)^3 + 43320a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 5214a'(t)^2a''(t)a^{(4)}(t)a(t)^3 + \\
& 2238a'(t)^5a^{(5)}(t)a(t)^3 + 462a'(t)^3a^{(5)}(t)a(t)^3 - 162162a'(t)^4a''(t)^3a(t)^2 - \\
& 11880a'(t)^2a''(t)^3a(t)^2 - 103884a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 13332a'(t)^3a''(t)a^{(3)}(t)a(t)^2 - \\
& 6138a'(t)^6a^{(4)}(t)a(t)^2 - 1287a'(t)^4a^{(4)}(t)a(t)^2 + 76440a'(t)^6a''(t)^2a(t) + \\
& 10428a'(t)^4a''(t)^2a(t) + 11700a'(t)^7a^{(3)}(t)a(t) + 2475a'(t)^5a^{(3)}(t)a(t) - 11700a'(t)^8a''(t) - \\
& 2475a'(t)^6a''(t)).
\end{aligned}$$

4. COMPUTATION OF THE TERM a_{12} IN THE EXPANSION OF THE SPECTRAL ACTION

We pursue the computation of the term a_{12} in the expansion of the spectral action for Robertson-Walker metrics by employing pseudodifferential calculus to find the term r_{12} for the parametrix of $\lambda - D^2$, which is homogeneous of order -14 , and by performing the appropriate integrations. Since there is no universal formula in the literature for this term, we have performed two heavy computations, one in Hopf coordinates and the other in spherical coordinates, to form a check on the validity of the outcome of our calculations. Another efficient way of computing the term a_{12} is to use the direct method of [9].

4.1. The result of the computation in Hopf coordinates. Continuing the recursive procedure commenced in the previous section and exploiting computer assistance, while the calculation becomes significantly heavier for the term a_{12} , we find the following expression:

$$\begin{aligned}
& a_{12} = \\
& \frac{1}{17297280a(t)^8} \left(3a^{(12)}(t)a(t)^{10} - 1057a^{(6)}(t)^2a(t)^9 - 1747a^{(5)}(t)a^{(7)}(t)a(t)^9 - \right. \\
& 970a^{(4)}(t)a^{(8)}(t)a(t)^9 - 317a^{(3)}(t)a^{(9)}(t)a(t)^9 - 34a''(t)a^{(10)}(t)a(t)^9 + \\
& 21a'(t)a^{(11)}(t)a(t)^9 + 5001a^{(4)}(t)^3a(t)^8 + 2419a''(t)a^{(5)}(t)^2a(t)^8 + \\
& 19174a^{(3)}(t)a^{(4)}(t)a^{(5)}(t)a(t)^8 + 4086a^{(3)}(t)^2a^{(6)}(t)a(t)^8 + 2970a''(t)a^{(4)}(t)a^{(6)}(t)a(t)^8 - \\
& 5520a'(t)a^{(5)}(t)a^{(6)}(t)a(t)^8 - 511a''(t)a^{(3)}(t)a^{(7)}(t)a(t)^8 - 4175a'(t)a^{(4)}(t)a^{(7)}(t)a(t)^8 - \\
& 745a''(t)^2a^{(8)}(t)a(t)^8 - 2289a'(t)a^{(3)}(t)a^{(8)}(t)a(t)^8 - 828a'(t)a''(t)a^{(9)}(t)a(t)^8 - \\
& 62a'(t)^2a^{(10)}(t)a(t)^8 - 13a^{(10)}(t)a(t)^8 + 45480a^{(3)}(t)^4a(t)^7 + 152962a''(t)^2a^{(4)}(t)^2a(t)^7 + \\
& 203971a'(t)a^{(3)}(t)a^{(4)}(t)^2a(t)^7 + 21369a'(t)^2a^{(5)}(t)^2a(t)^7 + 1885a^{(5)}(t)^2a(t)^7 + \\
& 410230a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^7 + 163832a'(t)a^{(3)}(t)^2a^{(5)}(t)a(t)^7 + \\
& 250584a''(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^7 + 244006a'(t)a''(t)a^{(4)}(t)a^{(5)}(t)a(t)^7 + \\
& 42440a''(t)^3a^{(6)}(t)a(t)^7 + 163390a'(t)a''(t)a^{(3)}(t)a^{(6)}(t)a(t)^7 + \\
& 35550a'(t)^2a^{(4)}(t)a^{(6)}(t)a(t)^7 + 3094a^{(6)}(t)a^{(6)}(t)a(t)^7 + 34351a'(t)a''(t)^2a^{(7)}(t)a(t)^7 + \\
& 19733a'(t)^2a^{(3)}(t)a^{(7)}(t)a(t)^7 + 1625a^{(3)}(t)a^{(7)}(t)a(t)^7 + 6784a'(t)^2a''(t)a^{(8)}(t)a(t)^7 + \\
& 520a''(t)a^{(8)}(t)a(t)^7 + 308a'(t)^3a^{(9)}(t)a(t)^7 + 52a'(t)a^{(9)}(t)a(t)^7 - \\
& 2056720a'(t)a''(t)a^{(3)}(t)^3a(t)^6 - 1790580a''(t)^3a^{(3)}(t)^2a(t)^6 - \\
& 900272a'(t)^2a''(t)a^{(4)}(t)^2a(t)^6 - 31889a''(t)a^{(4)}(t)^2a(t)^6 - 643407a''(t)^4a^{(4)}(t)a(t)^6 - \\
& 1251548a'(t)^2a^{(3)}(t)^2a^{(4)}(t)a(t)^6 - 43758a^{(3)}(t)^2a^{(4)}(t)a(t)^6 -
\end{aligned}$$

$$\begin{aligned}
& 4452042a'(t)a''(t)^2a^{(3)}(t)a^{(4)}(t)a(t)^6 - 836214a'(t)a''(t)^3a^{(5)}(t)a(t)^6 - \\
& 1400104a'(t)^2a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - 48620a''(t)a^{(3)}(t)a^{(5)}(t)a(t)^6 - \\
& 181966a'(t)^3a^{(4)}(t)a^{(5)}(t)a(t)^6 - 18018a'(t)a^{(4)}(t)a^{(5)}(t)a(t)^6 - \\
& 319996a'(t)^2a''(t)^2a^{(6)}(t)a(t)^6 - 11011a''(t)^2a^{(6)}(t)a(t)^6 - \\
& 115062a'(t)^3a^{(3)}(t)a^{(6)}(t)a(t)^6 - 11154a'(t)a^{(3)}(t)a^{(6)}(t)a(t)^6 - \\
& 42764a'(t)^3a''(t)a^{(7)}(t)a(t)^6 - 4004a'(t)a''(t)a^{(7)}(t)a(t)^6 - 1649a'(t)^4a^{(8)}(t)a(t)^6 - \\
& 286a'(t)^2a^{(8)}(t)a(t)^6 + 460769a''(t)^6a(t)^5 + 1661518a'(t)^3a^{(3)}(t)^3a(t)^5 + \\
& 83486a'(t)a^{(3)}(t)^3a(t)^5 + 13383328a'(t)^2a''(t)^2a^{(3)}(t)^2a(t)^5 + 222092a''(t)^2a^{(3)}(t)^2a(t)^5 + \\
& 342883a'(t)^4a^{(4)}(t)^2a(t)^5 + 36218a'(t)^2a^{(4)}(t)^2a(t)^5 + 7922361a'(t)a''(t)^4a^{(3)}(t)a(t)^5 + \\
& 6367314a'(t)^2a''(t)^3a^{(4)}(t)a(t)^5 + 109330a''(t)^3a^{(4)}(t)a(t)^5 + \\
& 7065862a'(t)^3a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + 360386a'(t)a''(t)a^{(3)}(t)a^{(4)}(t)a(t)^5 + \\
& 1918386a'(t)^3a''(t)^2a^{(5)}(t)a(t)^5 + 98592a'(t)a''(t)^2a^{(5)}(t)a(t)^5 + \\
& 524802a'(t)^4a^{(3)}(t)a^{(5)}(t)a(t)^5 + 55146a'(t)^2a^{(3)}(t)a^{(5)}(t)a(t)^5 + \\
& 226014a'(t)^4a''(t)a^{(6)}(t)a(t)^5 + 23712a'(t)^2a''(t)a^{(6)}(t)a(t)^5 + 8283a'(t)^5a^{(7)}(t)a(t)^5 + \\
& 1482a'(t)^3a^{(7)}(t)a(t)^5 - 7346958a'(t)^2a''(t)^5a(t)^4 - 72761a''(t)^5a(t)^4 - \\
& 11745252a'(t)^4a''(t)a^{(3)}(t)^2a(t)^4 - 725712a'(t)^2a''(t)a^{(3)}(t)^2a(t)^4 - \\
& 27707028a'(t)^3a''(t)^3a^{(3)}(t)a(t)^4 - 819520a'(t)a''(t)^3a^{(3)}(t)a(t)^4 - \\
& 8247105a'(t)^4a''(t)^2a^{(4)}(t)a(t)^4 - 520260a'(t)^2a''(t)^2a^{(4)}(t)a(t)^4 - \\
& 1848228a'(t)^5a^{(3)}(t)a^{(4)}(t)a(t)^4 - 205296a'(t)^3a^{(3)}(t)a^{(4)}(t)a(t)^4 - \\
& 973482a'(t)^5a''(t)a^{(5)}(t)a(t)^4 - 110136a'(t)^3a''(t)a^{(5)}(t)a(t)^4 - 36723a'(t)^6a^{(6)}(t)a(t)^4 - \\
& 6747a'(t)^4a^{(6)}(t)a(t)^4 + 17816751a'(t)^4a''(t)^4a(t)^3 + 721058a'(t)^2a''(t)^4a(t)^3 + \\
& 2352624a'(t)^6a^{(3)}(t)^2a(t)^3 + 274170a'(t)^4a^{(3)}(t)^2a(t)^3 + 24583191a'(t)^5a''(t)^2a^{(3)}(t)a(t)^3 + \\
& 1771146a'(t)^3a''(t)^2a^{(3)}(t)a(t)^3 + 3256248a'(t)^6a''(t)a^{(4)}(t)a(t)^3 + \\
& 389376a'(t)^4a''(t)a^{(4)}(t)a(t)^3 + 135300a'(t)^7a^{(5)}(t)a(t)^3 + 25350a'(t)^5a^{(5)}(t)a(t)^3 - \\
& 15430357a'(t)^6a''(t)^3a(t)^2 - 1252745a'(t)^4a''(t)^3a(t)^2 - 7747848a'(t)^7a''(t)a^{(3)}(t)a(t)^2 - \\
& 967590a'(t)^5a''(t)a^{(3)}(t)a(t)^2 - 385200a'(t)^8a^{(4)}(t)a(t)^2 - 73125a'(t)^6a^{(4)}(t)a(t)^2 + \\
& 5645124a'(t)^8a''(t)^2a(t) + 741195a'(t)^6a''(t)^2a(t) + 749700a'(t)^9a^{(3)}(t)a(t) + \\
& 143325a'(t)^7a^{(3)}(t)a(t) - 749700a'(t)^{10}a''(t) - 143325a'(t)^8a''(t)).
\end{aligned}$$

4.2. Agreement of the result with computations in spherical coordinates.

Taking a similar route as in §2, we explicitly write the Dirac operator for the Roberson-Walker metric in spherical coordinates

$$ds^2 = dt^2 + a^2(t) (d\chi^2 + \sin^2(\chi) (d\theta^2 + \sin^2(\theta) d\varphi^2)).$$

Using the computations carried out in [9] with the orthonormal coframe

$$dt, \quad a(t) d\chi, \quad a(t) \sin \chi d\theta, \quad a(t) \sin \chi \sin \theta d\varphi,$$

the corresponding matrix of connection 1-forms for the Levi-Civita connection is written as

$$\begin{pmatrix}
0 & -a'(t)d\chi & -a'(t)\sin(\chi)d\theta & -a'(t)\sin(\chi)\sin(\theta)d\varphi \\
a'(t)d\chi & 0 & -\cos(\chi)d\theta & -\cos(\chi)\sin(\theta)d\varphi \\
a'(t)\sin(\chi)d\theta & \cos(\chi)d\theta & 0 & -\cos(\theta)d\varphi \\
a'(t)\sin(\chi)\sin(\theta)d\varphi & \cos(\chi)\sin(\theta)d\varphi & \cos(\theta)d\varphi & 0
\end{pmatrix}.$$

Lifting to the spin bundle by means of the Lie algebra isomorphism $\mu : \mathfrak{so}(4) \rightarrow \mathfrak{spin}(4)$ and writing the formula for the Dirac operator yield the following expression

for this operator expressed in spherical coordiantes:

$$D = \gamma^1 \frac{\partial}{\partial t} + \gamma^2 \frac{1}{a} \frac{\partial}{\partial \chi} + \gamma^3 \frac{1}{a \sin \chi} \frac{\partial}{\partial \theta} + \gamma^4 \frac{1}{a \sin \chi \sin \theta} \frac{\partial}{\partial \varphi} \\ + \frac{3a'}{2a} \gamma^1 + \frac{\cot(\chi)}{a} \gamma^2 + \frac{\cot(\theta)}{2a \sin(\chi)} \gamma^3.$$

Thus the pseudodifferential symbol of D is given by

$$\sigma_D(x, \xi) = i\gamma^1 \xi_1 + \frac{i}{a} \gamma^2 \xi_2 + \frac{i}{a \sin(\chi)} \gamma^3 \xi_3 + \frac{i}{a \sin(\chi) \sin(\theta)} \gamma^4 \xi_4 \\ + \frac{3a'}{2a} \gamma^1 + \frac{\cot(\chi)}{a} \gamma^2 + \frac{\cot(\theta)}{2a \sin(\chi)} \gamma^3.$$

Accordingly, the symbol of D^2 is the sum $p'_2 + p'_1 + p'_0$ of three homogeneous components

$$p'_2 = \xi_1^2 + \frac{1}{a(t)^2} \xi_2^2 + \frac{1}{a(t)^2 \sin^2(\chi)} \xi_3^2 + \frac{1}{a(t)^2 \sin^2(\theta) \sin^2(\chi)} \xi_4^2, \\ p'_1 = -\frac{3ia'(t)}{a(t)} \xi_1 - \frac{i}{a(t)^2} (\gamma^{12} a'(t) + 2 \cot(\chi)) \xi_2 \\ - \frac{i}{a(t)^2} (\gamma^{13} \csc(\chi) a'(t) + \cot(\theta) \csc^2(\chi) + \gamma^{23} \cot(\chi) \csc(\chi)) \xi_3 \\ - \frac{i}{a(t)^2} (\csc(\theta) \csc(\chi) a'(t) \gamma^{14} + \cot(\theta) \csc(\theta) \csc^2(\chi) \gamma^{34} \\ + \csc(\theta) \cot(\chi) \csc(\chi) \gamma^{24}) \xi_4, \\ p'_0 = \frac{1}{8a(t)^2} (-12a(t)a''(t) - 6a'(t)^2 + 3 \csc^2(\theta) \csc^2(\chi) - \cot^2(\theta) \csc^2(\chi) + \\ 4i \cot(\theta) \cot(\chi) \csc(\chi) - 4i \cot(\theta) \cot(\chi) \csc(\chi) - 4 \cot^2(\chi) + 5 \csc^2(\chi) + 4) \\ - \frac{(\cot(\theta) \csc(\chi) a'(t))}{2a(t)^2} \gamma^{13} - \frac{(\cot(\chi) a'(t))}{a(t)^2} \gamma^{12} - \frac{(\cot(\theta) \cot(\chi) \csc(\chi))}{2a(t)^2} \gamma^{23}.$$

We have performed the computation of the heat coefficients up to the term a_{12} using the latter symbols and have checked the agreement of the result with the computations in Hopf coordinates, presented in the previous subsections. This is in particular of great importance for the term a_{12} , since it ensures the validity of our computations performed in two different coordinates.

4.3. Agreement with the full expansion for the round metric. We first recall the full expansion for the spectral action for the round metric, namely the case $a(t) = \sin(t)$, worked out in [9]. Then we show that the term a_{12} presented in §4.1 reduces correctly to the round case.

The method devised in [9] has wide applicability in the spectral action computations since it can be used for the cases when the eigenvalues of the square of the Dirac operator have a polynomial expression while their multiplicities are also given by polynomials. In the case of the round metric on \mathbb{S}^4 , after remarkable computations based on the Euler-Maclaurin formula, this method leads to the following

expression with control over the remainder term [9]:

$$\begin{aligned} \frac{3}{4} \text{Trace}(f(tD^2)) &= \int_0^\infty f(tx^2)(x^3 - x)dx + \frac{11f(0)}{120} - \frac{31f'(0)t}{2520} + \frac{41f''(0)t^2}{10080} \\ &\quad - \frac{31f^{(3)}(0)t^3}{15840} + \frac{10331f^{(4)}(0)t^4}{8648640} - \frac{3421f^{(5)}(0)t^5}{3931200} + \cdots + R_m. \end{aligned}$$

This implies that the term a_{12} in the expansion of the spectral action for the round metric is equal to $\frac{10331}{6486480}$. To check our calculations against this result, we find that for $a(t) = \sin(t)$ the expression for $a_{12}(t)$ reduces to $\frac{10331 \sin^3(t)}{8648640}$, and hence

$$a_{12} = \int_0^\pi a_{12}(\mathbb{S}^4) dt = \frac{4}{3} \frac{10331}{8648640} = \frac{10331}{6486480},$$

which is in complete agreement with the result in [9], mentioned above.

5. CHAMESEDDINE-CONNES' CONJECTURE

In this section we prove a conjecture of Chamseddine and Connes from [9]. More precisely, we show that the term a_{2n} in the asymptotic expansion of the spectral action for Robertson-Walker metrics is, up to multiplication by $a(t)^{3-2n}$, of the form $Q_{2n}(a, a', \dots, a^{(2n)})$, where Q_{2n} is a polynomial with rational coefficients.

5.1. Proof of rationality of the coefficients in the expressions for a_{2n} . A crucial point that enables us to furnish the proof of our main theorem, namely the proof of the conjecture mentioned above, is the independence of the integral kernel of the heat operator of the Dirac operator of the Robertson-Walker metric from the variables ϕ_1, ϕ_2, η . Note that since the symbol and the metric are independent of ϕ_1, ϕ_2 , the computations involved in the symbol calculus clearly imply the independence of the terms e_n from these variables. However, the independence of e_n from η is not evident, which is proved as follows.

Lemma 5.1. *The heat kernel $k(t, x, x)$ for the Robertson-Walker metric is independent of ϕ_1, ϕ_2, η .*

Proof. The round metric on \mathbb{S}^3 is the bi-invariant metric on $\text{SU}(2)$ induced from the Killing form of its Lie algebra $\mathfrak{su}(2)$. The corresponding Levi-Civita connection restricted to the left invariant vector fields is given by $\frac{1}{2}[X, Y]$, and to the right invariant vector fields by $\frac{-1}{2}[X, Y]$. Since the Killing form is ad-invariant, we have

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \quad X, Y, Z \in \mathfrak{su}(2),$$

which implies that in terms of the connection on left (right) invariant vector fields X, Y, Z , it can be written as

$$(9) \quad \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0.$$

Considering the fact that $\nabla X : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is an endomorphism of the tangent bundle, the latter identity holds for any $Y, Z \in \mathfrak{X}(M)$. Therefore, the equation (9) is the Killing equation and shows that any left and right invariant vector field on $\text{SU}(2)$ is a Killing vector field.

By direct computation in Hopf coordinates, we find the following vector fields which respectively form bases for left and right invariant vector fields on $SU(2)$:

$$\begin{aligned}
X_1^L &= \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^L &= \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \eta} + \cot(\eta) \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_1} - \tan(\eta) \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_2}, \\
X_3^L &= \cos(\phi_1 + \phi_2) \frac{\partial}{\partial \eta} - \cot(\eta) \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_1} + \tan(\eta) \sin(\phi_1 + \phi_2) \frac{\partial}{\partial \phi_2}, \\
X_1^R &= -\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}, \\
X_2^R &= -\sin(\phi_1 - \phi_2) \frac{\partial}{\partial \eta} - \cot(\eta) \cos(\phi_1 - \phi_2) \frac{\partial}{\partial \phi_1} - \tan(\eta) \cos(\phi_1 - \phi_2) \frac{\partial}{\partial \phi_2}, \\
X_3^R &= \cos(\phi_1 - \phi_2) \frac{\partial}{\partial \eta} - \cot(\eta) \sin(\phi_1 - \phi_2) \frac{\partial}{\partial \phi_1} - \tan(\eta) \sin(\phi_1 - \phi_2) \frac{\partial}{\partial \phi_2}.
\end{aligned}$$

One can check that these vector fields are indeed Killing vector fields for the Robertson-Walker metrics on the four dimensional space. Thus, for any isometry invariant function f we have:

$$\begin{aligned}
\frac{\partial}{\partial \phi_1} f &= \frac{1}{2} (X_1^L - X_1^R) f = 0, \\
\frac{\partial}{\partial \phi_2} f &= \frac{1}{2} (X_1^L + X_1^R) f = 0, \\
\frac{\partial}{\partial \eta} f &= (\sin(\phi_1 + \phi_2) X_2^L + \cos(\phi_1 + \phi_2) X_3^L) f = 0.
\end{aligned}$$

In particular, the heat kernel restricted to the diagonal, $k(t, x, x)$, is independent of ϕ_1, ϕ_2, η , and so are the coefficient functions e_n in its asymptotic expansion. \square

We stress that although $e_n(x)$ is independent of η, ϕ_1, ϕ_2 , its components denoted by $e_{n,j,\alpha}$ in the proof of the following theorem are not necessarily independent of these variables.

Theorem 5.1. *The term a_{2n} in the expansion of the spectral action for the Robertson-Walker metric with cosmic scale factor $a(t)$ is of the form*

$$\frac{1}{a(t)^{2n-3}} Q_{2n} \left(a(t), a'(t), \dots, a^{(2n)}(t) \right),$$

where Q_{2n} is a polynomial with rational coefficients.

Proof. Using (7) we can write

$$(10) \quad e_n = \sum_{\substack{2j-2-|\alpha|=n \\ n/2+1 \leq j \leq 2n+1}} c_\alpha e_{n,j,\alpha},$$

where

$$e_{n,j,\alpha} = \frac{1}{(j-1)!} r_{n,j,\alpha} a(t)^{\alpha_2+\alpha_3+\alpha_4} \sin(\eta)^{\alpha_3} \cos(\eta)^{\alpha_4}.$$

The recursive equation (8) implies that

$$(11) \quad e_{n,j,\alpha} =$$

$$\begin{aligned}
& \frac{1}{(j-1)a(t)} \left((\gamma^{14}a'(t) - \tan(\eta)\gamma^{24})e_{n-1,j-1,\alpha-\mathbf{e}_4} + (\gamma^{13}a'(t) + \cot(\eta)\gamma^{23})e_{n-1,j-1,\alpha-\mathbf{e}_3} + \right. \\
& (\gamma^{12}a'(t) + 1((2\alpha_4 - 1)\tan(\eta) + (1 - 2\alpha_3)\cot(\eta)))e_{n-1,j-1,\alpha-\mathbf{e}_2} + \\
& 4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_2} + 4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_3} + 4a'(t)e_{n-1,j-2,\alpha-\mathbf{e}_1-2\mathbf{e}_4} + (-2\alpha_2 - \\
& 2\alpha_3 - 2\alpha_4 + 3)a'(t)e_{n-1,j-1,\alpha-\mathbf{e}_1} + 2a(t)\frac{\partial}{\partial t}e_{n-1,j-1,\alpha-\mathbf{e}_1} - 4\tan(\eta)e_{n-1,j-2,\alpha-\mathbf{e}_2-2\mathbf{e}_4} + \\
& \left. 4\cot(\eta)e_{n-1,j-2,\alpha-\mathbf{e}_2-2\mathbf{e}_3} + 2\frac{\partial}{\partial \eta}e_{n-1,j-1,\alpha-\mathbf{e}_2} \right) \\
& + \frac{1}{(j-1)a(t)^2} \left(a(t)^2\frac{\partial^2}{\partial t^2}e_{n-2,j-1,\alpha} + 4a'(t)a(t)\frac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_2} + 4a'(t)a(t)\frac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_3} + \right. \\
& 4a'(t)a(t)\frac{\partial}{\partial t}e_{n-2,j-2,\alpha-2\mathbf{e}_4} + (-2\alpha_2 - 2\alpha_3 - 2\alpha_4 + 3)a'(t)a(t)\frac{\partial}{\partial t}e_{n-2,j-1,\alpha} + \\
& 4a'(t)^2e_{n-2,j-3,\alpha-4\mathbf{e}_2} + 8a'(t)^2e_{n-2,j-3,\alpha-2\mathbf{e}_2-2\mathbf{e}_3} + 8a'(t)^2e_{n-2,j-3,\alpha-2\mathbf{e}_2-2\mathbf{e}_4} + \\
& 4\cot(\eta)\frac{\partial}{\partial \eta}e_{n-2,j-2,\alpha-2\mathbf{e}_3} - 4\tan(\eta)\frac{\partial}{\partial \eta}e_{n-2,j-2,\alpha-2\mathbf{e}_4} + \frac{\partial^2}{\partial \eta^2}e_{n-2,j-1,\alpha} + (2\cot(\eta)\gamma^{12}a'(t) + \\
& (-4(\alpha_2 + \alpha_3 + \alpha_4 - 2)a'(t)^2 + 4(-(\alpha_3 - 1)\csc^2(\eta) + \alpha_3 + \alpha_4 - 2) + 2a(t)a''(t)))e_{n-2,j-2,\alpha-2\mathbf{e}_3} + \\
& ((\cot(\eta)(1 - 2\alpha_3) + (2\alpha_4 - 1)\tan(\eta)) + \gamma^{12}a'(t))\frac{\partial}{\partial \eta}e_{n-2,j-1,\alpha} + ((-4(\alpha_2 + \alpha_3 + \alpha_4 - 2)a'(t)^2 + \\
& 4(-(\alpha_4 - 1)\sec^2(\eta) + \alpha_3 + \alpha_4 - 2) + 2a(t)a''(t)) - 2\gamma^{12}\tan(\eta)a'(t))e_{n-2,j-2,\alpha-2\mathbf{e}_4} + 8(a'(t)^2 - \\
& 1)e_{n-2,j-3,\alpha-2\mathbf{e}_4} + 4(\cot^2(\eta) + a'(t)^2)e_{n-2,j-3,\alpha-4\mathbf{e}_3} + 4(\tan^2(\eta) + a'(t)^2)e_{n-2,j-3,\alpha-4\mathbf{e}_4} + \\
& (2a(t)a''(t) - 4(\alpha_2 + \alpha_3 + \alpha_4 - 2)a'(t)^2)e_{n-2,j-2,\alpha-2\mathbf{e}_2} + (\frac{1}{2}(\cot(\eta)(1 - 2\alpha_3) + (2\alpha_4 - \\
& 1)\tan(\eta))\gamma^{12}a'(t) + \frac{1}{4}((\alpha_3^2 - 1)\csc^2(\eta) - 4(\alpha_3 + \alpha_4 - 1)^2 + (2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 3)(2\alpha_2 + \\
& 2\alpha_3 + 2\alpha_4 - 1)a'(t)^2 + \sec^2(\eta)(4\alpha_4^2 - 1) - 2(2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 3)a(t)a''(t)))e_{n-2,j-1,\alpha} \Big).
\end{aligned}$$

The functions associated with the initial indices are:

$$\begin{aligned}
e_{0,1,0,0,0} &= 1, & e_{1,2,1,0,0,0} &= \frac{3ia'(t)}{a(t)}, & e_{1,3,1,2,0,0} &= \frac{2ia'(t)}{a(t)}, \\
e_{1,3,1,0,2,0} &= \frac{2ia'(t)}{a(t)}, & e_{1,3,1,0,0,2} &= \frac{2ia'(t)}{a(t)}, & e_{1,3,0,1,0,2} &= -\frac{(2i)\tan(\eta)}{a(t)}, \\
e_{1,3,0,1,2,0} &= \frac{(2i)\cot(\eta)}{a(t)}, & e_{1,2,0,0,1,0} &= \frac{i\gamma^{13}a'(t)}{a(t)} + \frac{i\gamma^{23}\cot(\eta)}{a(t)}, \\
e_{1,2,0,0,0,1} &= \frac{i\gamma^{14}a'(t)}{a(t)} - \frac{i\gamma^{24}\tan(\eta)}{a(t)}, & e_{1,2,0,1,0,0} &= \frac{2i\cot(2\eta)}{a(t)} + \frac{i\gamma^{12}a'(t)}{a(t)}.
\end{aligned}$$

It is then apparent that e_0 and e_1 are, respectively, a polynomial in $a(t)$, and a polynomial in $a(t)$ and $a'(t)$, divided by some powers of $a(t)$. Thus, it follows from the above recursive formula that all $e_{n,j,\alpha}$ are of this form. Accordingly, we have

$$e_n = \frac{P_n}{a(t)^{d_n}},$$

where P_n is a polynomial in $a(t)$ and its derivatives with matrix coefficients. Writing $e_{n,j,\alpha} = P_{n,j,\alpha}/a(t)^{d_n}$, we obtain $d_n = \max\{d_{n-1} + 1, d_{n-2} + 2\}$. Starting with $d_0 = 0$, $d_1 = -1$, and following to obtain $d_n = n$, we conclude that

$$e_{n,j,\alpha} = \frac{1}{a^n(t)} P_{n,j,\alpha}(a(t), \dots, a^{(n)}(t)),$$

where $P_{n,j,\alpha}$ is a polynomial whose coefficients are matrices with entries in the algebra generated by $\sin(\eta)$, $\csc(\eta)$, $\cos(\eta)$, $\sec(\eta)$ and rational numbers.

In the calculation of the even terms a_{2n} , only even α_k have contributions in the summation (10). This implies that the corresponding c_α is a rational multiple of π^2 and P_{2n} is a polynomial with rational matrix coefficients, which is independent of variables η, ϕ_1, ϕ_2 by Lemma 5.1. Hence

$$a_{2n} = \frac{1}{16\pi^4} \int_{\mathbb{S}_a^3} \text{tr}(e_{2n}) d\text{vol}_g = \frac{2\pi^2 a(t)^3}{16\pi^4} \text{tr}\left(\frac{P_{2n}}{a(t)^{2n}}\right) = \frac{Q_{2n}}{a(t)^{n-3}},$$

where Q_{2n} is a polynomial in $a(t), a'(t), \dots, a^{(2n)}(t)$ with rational coefficients. \square

The polynomials $P_{n,j,\alpha}$ also satisfy recursive relations that illuminate interesting features about their structure.

Proposition 5.1. *Each $P_{n,j,\alpha}$ is a finite sum of the form*

$$\sum c_k a(t)^{k_0} a'(t)^{k_1} \dots a^{(n)}(t)^{k_n},$$

where each c_k is a matrix of functions that are independent from the variable t , and $\sum_{j=0}^n k_j = \sum_{j=0}^n j k_j = l$, for some $0 \leq l \leq n$.

Proof. This follows from an algebraically lengthy recursive formula for $P_{n,j,\alpha}$ which stems from the equation (8), similar to the recursive formula for $e_{n,j,\alpha}$ in the proof of Theorem 5.1. In addition, one needs to find the following initial cases:

$$\begin{aligned} P_{0,1,0,0,0,0} &= I, & P_{1,2,1,0,0,0} &= 3ia'(t), & P_{1,2,0,0,1,0} &= i\gamma^{13}a'(t) + i\gamma^{23}\cot(\eta), \\ P_{1,2,0,0,0,1} &= i\gamma^{14}a'(t) - i\gamma^{24}\tan(\eta), & P_{1,2,0,1,0,0} &= 2i\cot(2\eta) + i\gamma^{12}a'(t), \\ P_{1,3,0,1,0,2} &= -2i\tan(\eta), & P_{1,3,0,1,2,0} &= 2i\cot(\eta), & P_{1,3,1,2,0,0} &= 2ia'(t), \\ P_{1,3,1,0,2,0} &= 2ia'(t), & P_{1,3,1,0,0,2} &= 2ia'(t). \end{aligned}$$

\square

5.2. A recursive formula for the coefficient of the highest order term in a_{2n} . The highest derivative of the cosmic scale factor $a(t)$ in the expression for a_n is seen in the term $a(t)^{n-1}a^{(n)}(t)$, which has a rational coefficient based on Theorem 5.1. Let us denote the coefficient of $a(t)^{n-1}a^{(n)}(t)$ in a_n by h_n . Since the coefficients h_n are limited to satisfy the recursive relations derived in the proof of the following proposition, one can find the following closed formula for these coefficients.

Proposition 5.2. *The coefficient h_n of $a(t)^{n-1}a^{(n)}(t)$ in a_n is equal to*

$$\sum_{\substack{[n/2] + 1 \leq j \leq 2n+1 \\ 0 \leq k \leq j - n/2 - 1}} \Gamma\left(\frac{2k+1}{2}\right) H_{n,j,2k},$$

where, starting from

$$\begin{aligned} H_{1,2,1} &= H_{1,3,1} = \frac{3i}{2\sqrt{\pi}}, & H_{2,4,2} &= -\frac{1}{\sqrt{\pi}}, \\ H_{2,3,0} &= H_{2,2,0} = \frac{3}{4\sqrt{\pi}}, & H_{2,3,2} &= -\frac{3}{2\sqrt{\pi}}, \end{aligned}$$

the quantities $H_{n,j,\alpha}$ are computed recursively by

$$H_{n,j,\alpha} = \frac{1}{j-1} (H_{n-2,j-1,\alpha} + 2iH_{n-1,j-1,\alpha-1}).$$

Proof. It follows from Proposition 5.1 that the highest derivative of $a(t)$ in a_n appears in the term $a(t)^{n-1}a^{(n)}(t)$. By a careful analysis of the equation (11) we find that only the terms

$$\frac{1}{j-1} \left(a(t)^2 \frac{\partial^2}{\partial t^2} P_{n-2,j-1,\alpha} + 2ia(t) \frac{\partial}{\partial t} P_{n-1,j-1,\alpha-\mathbf{e}_1} \right)$$

contribute to its recursive formula. Denoting the corresponding monomial in $P_{n,j,\alpha}$ by $H_{n,j,\alpha}a(t)^{n-1}a^{(n)}(t)$ and substituting it into the above formula we obtain the equation

$$H_{n,j,\alpha} = \frac{1}{j-1}(H_{n-2,j-1,\alpha} + 2iH_{n-1,j-1,\alpha-\mathbf{e}_1}),$$

for any $n > 2$. Denoting

$$H_{n,j,\alpha_1} = \sum \prod_{k=2}^4 \Gamma\left(\frac{\alpha_k+1}{2}\right) \frac{(-1)^{\alpha_k}+1}{2} \operatorname{tr} \left(\frac{1}{(2\pi)^2} \int_0^{\pi/2} H_{n,j,\alpha_1,\alpha_2,\alpha_3,\alpha_4} d\eta \right),$$

the recursive formula converts to

$$H_{n,j,\alpha} = \frac{1}{j-1}(H_{n-2,j-1,\alpha} + 2iH_{n-1,j-1,\alpha-1}).$$

Thus, the coefficient of $a(t)^{n-1}a^{(n)}(t)$ in a_n is given by the above expression. \square

Using the above proposition we find that:

$$\begin{aligned} h_2 &= \frac{1}{4}, & h_4 &= \frac{1}{40}, & h_6 &= \frac{1}{560}, & h_8 &= \frac{1}{10080}, & h_{10} &= \frac{1}{221760}, \\ h_{12} &= \frac{1}{5765760}, & h_{14} &= \frac{1}{172972800}, & h_{16} &= \frac{1}{5881075200}, \\ h_{18} &= \frac{1}{223480857600}, & h_{20} &= \frac{1}{9386196019200}. \end{aligned}$$

6. CONCLUSIONS

Pseudodifferential calculus is an effective tool for applying heat kernel methods to compute the terms in the expansion of a spectral action. We have used this technique to derive the terms up to a_{12} in the expansion of the spectral action for the Robertson-Walker metric on a 4-dimensional geometry with a general cosmic scale factor $a(t)$. Performing the computations in Hopf coordinates, which reflects the symmetry of the space more conveniently at least from a technical point of view, we proved the independence of the integral kernel of the corresponding heat operator from three coordinates of the space. This allowed us to furnish the proof of the conjecture of Chamseddine and Connes on rationality of the coefficients of the polynomials in $a(t)$ and its derivatives that describe the general terms a_{2n} in the expansion.

The terms up to a_{10} were previously computed in [9] using their direct method, where the terms up to a_6 were checked against Gilkey's universal formulas [17, 18]. The outcome of our computations confirms the previously computed terms. Thus, we have formed a check on the terms a_8 and a_{10} . In order to confirm our calculation for the term a_{12} , we have performed a completely different computation in spherical coordinates and checked its agreement with our calculation in Hopf coordinates. It is worth emphasizing that the high complexity of the computations, which is overcome by computer assistance, raises the need to derive the expressions at least in two different ways to ensure their validity.

We have found a formula for the coefficient of the term with the highest derivative of $a(t)$ in a_{2n} for all n and make the following observation. The polynomials Q_{2n} in $a_{2n} = Q_{2n}(a(t), a'(t), \dots, a^{(2n)}(t)) / a(t)^{2n-3}$ are of the following form up to Q_{12} :

$$Q_{2n}(x_0, x_1, \dots, x_{2n}) = \sum_{17} c_k x_0^{k_0} x_1^{k_1} \dots x_{2n}^{k_{2n}}, \quad c_k \neq 0,$$

where the summation is over all tuples of non-negative integers $k = (k_0, k_1, \dots, k_{2n})$ such that either $\sum k_j = 2n$ while $\sum jk_j = 2n$, or $\sum k_j = 2n - 2$ while $\sum jk_j = 2n - 2$. This provides enough evidence and hope to shed more light on general structure of the terms a_{2n} by further investigations, which are under way.

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